Moderne Quantenfeldtheorie und Einführung in das Standardmodell

(exercise sheet 4)

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1. Uniqueness of the standard model scalar potential (2 points)

The usual $SU(2)_L \times U(1)_Y$ scalar potential for the standard model is of the form

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \,. \tag{1}$$

In principle one can also have $SU(2)_L \times U(1)_Y$ quartic invariants of the form

$$V_1(\Phi) = \lambda_1 (\Phi^\dagger \vec{\tau} \Phi)^T (\Phi^\dagger \vec{\tau} \Phi), \qquad V_2(\Phi) = \lambda_2 \, (\Phi^\dagger \tau^i \tau^j \Phi)(\Phi^\dagger \tau^i \tau^j \Phi), \tag{2}$$

where $\tau^i = \sigma^i/2$. Argue that those terms are really invariant under the SM gauge group and show that they can be reduced to the terms already present in the SM potential.

2. Two Higgs doublet model (1+1+3+4+1 points)

Suppose the Higgs doublet of the Standard Model (SM), here denoted as $\Phi_1(x)$ for convenience, is supplemented by a second complex doublet, $\Phi_2(x)$, transforming as $(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$ under $SU(3)_c \times SU(2)_L \times U(1)_Y$.

a) We can parametrize the second Higgs doublet as $\Phi_2 = \begin{pmatrix} \varphi_1 & \varphi_2 \end{pmatrix}^T$, where φ_1 and φ_2 denote complex scalar fields. What are the electric charges of φ_1 and φ_2?

b) Write out the covariant derivative $D_\mu \Phi_2$ explicitly in terms of the gauge fields G_μ^α, W_μ^a and B_μ.

c) Assuming that the potential is only a function of the invariants $\Phi_1^\dagger \Phi_1$, $\Phi_2^\dagger \Phi_2$ and $\Phi_1^T \epsilon \Phi_2$, where $\epsilon = i\sigma^2$. What is the most general renormalizable form of the potential? How many independent parameters does it contain? Do the parameters appearing in the potential need to be real? Are the combinations $\Phi_1^\dagger \Phi_2$ and $|\Phi_1^\dagger \Phi_2|^2$ invariant under the SM gauge group?

d) Suppose the parameters of the potential are such that it is minimized for

$$\langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} u + iw \\ 0 \end{pmatrix}, \tag{3}$$

where $u, v, w \in \mathbb{R}$. Do these values break the electromagnetic group $U(1)_{\text{em}}$ generated by the electric charge $Q = T_3 + Y$? Consider the kinetic Lagrangian

$$\mathcal{L}_{\text{kin}} = (D_\mu \Phi_1)^\dagger (D^\mu \Phi_1) + (D_\mu \Phi_2)^\dagger (D^\mu \Phi_2), \tag{4}$$

and calculate the masses of the gauge fields using the vacuum expectation values in (3). Call the mass eigenstates $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$, $Z_\mu = W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W$, and $A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W$, where $\cos\theta_W = g_2/\sqrt{g_1^2+g_2^2}$ and $e = g_1\cos\theta_W = g_2 \sin\theta_W$ (g_1, g_2 denote the couplings of B_μ and W^a_μ, respectively). Is the SM mass relation $M_W = M_Z \cos\theta_W$ also valid in this model?

Hint: It can be helpful to express the covariant derivative in terms of the mass eigenstates first. One should arrive at $D_\mu = \partial_\mu - ie\,QA_\mu - i\frac{g_2}{\cos\theta_W}(\tau^3 - Q\sin^2\theta_W)Z_\mu - i\frac{g_2}{\sqrt{2}}(W^+_\mu\tau^+ + W^-_\mu\tau^-)$, *where* Q *is the electric charge and* $\tau^\pm = \tau^1 \pm i\tau^2$ *with* $\tau^i = \sigma^i/2$.

e) What are the possible Yukawa couplings of the two scalar doublets, Φ_1 and Φ_2, to the quarks in the SM?

3. Custodial SU(2) symmetry (1+1+1+1 points)

In the Standard Model the W- and Z-boson masses are related by

$$\frac{M_W}{M_Z} = \frac{g_2}{\sqrt{g_1^2+g_2^2}} = \cos\theta_W, \tag{5}$$

where θ_W is called the Weinberg angle. In this exercise we want to investigate how this relation is connected to the details of the symmetry breaking sector. Assuming that a local gauge symmetry $SU(2)_L \times U(1)_Y$, where three of the gauge bosons transform in a triplet representation of $SU(2)$, gets spontaneously broken to $U(1)_{\mathrm{em}}$, the mass matrix has the following form ($M_W, M_3, M_0, m \in \mathbb{R}^+$)

$$\mathcal{M} = \begin{pmatrix} M_W^2 & 0 & 0 & 0 \\ 0 & M_W^2 & 0 & 0 \\ 0 & 0 & M_3^2 & m^2 \\ 0 & 0 & m^2 & M_0^2 \end{pmatrix}, \tag{6}$$

with the additional condition that one eigenvalue of the lower right 2×2-submatrix must vanish if we want to allow for a massless gauge boson after SSB.

a) Show that the last condition implies $m^2 = M_0 M_3$.

b) The eigenvector belonging to the zero eigenvalue shall be parametrized by $\vec{e} = (-\sin\theta_W, \cos\theta_W)^T$. Find the corresponding relations between θ_W, m and M_3.

c) The nonzero eigenvalue is just the Z-boson mass M_Z. Show that $M_3 = M_W$.

d) Let us now consider the Higgs doublet $\Phi(x)$. This field contains 4 real parameters, that can also be parametrized by a 4-vector $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, where each component field is a real scalar. Show that the kinetic term is invariant under a global $O(4)$ transformation of the fields. Is the SM Higgs potential invariant, too?

Consider the case that the Higgs field takes the following vacuum expectation value $\langle\Phi\rangle = \frac{1}{\sqrt{2}}(v,0,0,0)^T$, which breaks the $O(4)$-symmetry of the potential to an $O(3)$-symmetry. The remaining $\mathcal{O}(3)$-symmetry implies that the upper left 3×3-submatrix of $\mathcal{M}$ is proportional to the unit matrix. Since the group $O(3)$ is locally isomorphic to $SU(2)$, we speak of a custodial $SU(2)$ invariance of the symmetry breaking sector.

Please write down how long it took you to solve the exercises.