Moderne Quantenfeldtheorie und Einführung in das Standardmodell

(exercise sheet 3)

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1. Some group theory (1+4 points)

Let G be a compact Lie group with hermitian generators $T_a \in \mathrm{Lie}(G)$ for $a = 1, ..., d_G$, where d_G is the dimension of the group, that span the space of infinitesimal group transformations, i.e. the vector space of the Lie algebra $\mathrm{Lie}(G)$.

a) Part of the definition of a Lie algebra $\mathrm{Lie}(G)$ is the existence of a map $[.,.] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$, which is usually expressed by

$$[T_a, T_b] = i f_{abc} T_c \,, \tag{1}$$

where f_{abc} are the structure constants and a summation over c is understood. One further requires that the map is linear, antisymmetric and fulfills the Jacobi identity

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \,. \tag{2}$$

Show that this implies the following properties of the structure constants

$$f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} = 0 \,. \tag{3}$$

b) To give a finite-dimensional unitary representation of G , where d_r is the dimension of the representation, one has to find a set of $d_r \times d_r$ hermitian matrices T_r^a, $a = 1, ..., d_G$ that satisfy the defining relation (1) and act by multiplication on an d_r-complex dimensional vector $\vec{\phi} \in \mathbb{C}^{d_r}$ with components ϕ_i, $i = 1, ..., d_r$. In this case the bilinear $[.,.]$ in (1) is just the commutator of two $d_r \times d_r$ matrices. For an irreducible unitary representation one can normalize the generators T_r^a such that

$$\mathrm{Tr}(T_r^a T_r^b) = C(r)\, \delta^{ab} \,, \tag{4}$$

where $C(r)$ is a constant for each representation and is called the Dynkin index. Use this relation to show that

$$f^{abc} = -\frac{i}{C(r)} \mathrm{Tr}\left([T_r^a, T_r^b]\, T_r^c\right) , \tag{5}$$

and deduce that f_{abc} is antisymmetric in all indices. Check that the quadratic Casimir operator defined by $T_r^2 \equiv T_r^a T_r^a$ is invariant under the Lie algebra. Thus the matrix representation is proportional to

$$T_r^a T_r^a = C_2(r) \mathbf{1} \,, \tag{6}$$

where $\mathbf{1}$ is a $d_r \times d_r$ unit matrix and $C_2(r)$ is a constant for each representation. Derive a relation between $C(r)$ and $C_2(r)$. In case of the special unitary group $G = SU(N)$, one usually chooses the normalization $C(r) = 1/2$ for the fundamental representation. Derive for this choice the Casimir operator $C_2(r)$ in dependence of N.

2. Spontaneous symmetry breaking (SSB) (3+3+3 points)

In this exercise we want to investigate the relation between the spectrum of a theory before and after SSB, and how it depends on the (internal) symmetry property of the corresponding Lagrangian.

a) Consider a real scalar field $\phi(x)$ with the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)\,, \qquad\qquad V(\phi) = -\mu^2\phi^2 + \lambda\phi^4\,, \tag{7}$$

where stability requires $\lambda > 0$. First, under which internal symmetry of ϕ is the Lagrangian invariant? Then calculate all (classical) ground states $\langle\phi\rangle$ by minimizing the potential for both choices $\mu^2 \le 0$ and $\mu^2 > 0$. Now, consider the case $\mu^2 > 0$ and expand around the corresponding ground state with an excitation $\eta(x)$, i.e. $\phi(x) = \frac{v}{\sqrt{2}} + \eta(x)$, where v is called the vacuum expectation value (vev) and is defined via $\langle\phi\rangle = v/\sqrt{2}$. What is the spectrum of the Lagrangian, i.e. analyze the terms quadratic in the fields and count the degrees of freedom? What was the spectrum before SSB?

b) Now, we turn to a complex scalar field $\phi(x)$ with the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - V(\phi), \qquad\qquad V(\phi) = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \tag{8}$$

which is symmetric under a global U(1) transformation, i.e. $\phi(x) \to e^{-i\alpha}\phi(x)$ for $\alpha \in \mathbb{R}$. Calculate the ground states for $\mu^2 > 0$. Choose one ground state $\langle\phi\rangle = v/\sqrt{2}$ and consider a perturbation around it, i.e $\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x))$, where $\eta(x)$, $\xi(x)$ are real scalar fields. What is the spectrum of the Lagrangian? What was the spectrum before SSB?

c) This time, we promote the symmetry to a local U(1) gauge symmetry, thus

$$\mathcal{L} = (D_\mu \phi)^*(D^\mu \phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi), \qquad V(\phi) = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \tag{9}$$

where $D^\mu = \partial^\mu + ieA^\mu$ is the covariant derivative and $F_{\mu\nu}$ the field strength tensor. Consider the case of SSB with $\mu^2 > 0$ and choose the ground state $\langle\phi\rangle = v/\sqrt{2}$ with $v = \mu/\sqrt{\lambda}$. If we insert the same expansion as used in b), we could not easily read off the spectrum of the theory. Therefore we first define the following expansion

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))\, e^{i\xi(x)/v}\;, \tag{10}$$

which coincides with the one used in b) at leading order in the fields. Now, perform a suitable gauge transformation that eliminates the field $\xi(x)$ from the Lagrangian. What is the spectrum of the Lagrangian? What was the spectrum before SSB?

Please write down how long it took you to solve the exercises.