Moderne Quantenfeldtheorie und Einführung in das Standardmodell

(exercise sheet 2)

Matthias Neubert & Raoul Malm WS 2013/14

Hand-in on November 14th, 2013

1. Spin-1 propagators and gauge-fixing (2+2+3+2 points)

Consider the following action

$$\mathcal{S}_0 = -\frac{1}{4}\int d^4x \, F_{\mu\nu}(x)F^{\mu\nu}(x) \qquad \text{with} \qquad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \, . \tag{1}$$

a) Show that in momentum space it reads

$$\mathcal{S}_0 = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\tilde{A}^\mu(p)\mathcal{O}_{\mu\nu}\tilde{A}^\nu(-p) \quad \text{with} \quad \mathcal{O}_{\mu\nu}(p) = -\left(\eta_{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)p^2, \tag{2}$$

where $\tilde{A}_\mu(p)$ denotes the Fourier-transform of $A_\mu(x)$.

b) Consider the operators (transversal and longitudinal)

$$P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad\qquad P^L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}, \tag{3}$$

and show, that they are generalized projection operators in the sense, that

$$P^T_{\mu\nu}P^{T\nu\lambda} = P^{T\lambda}_{\mu}, \quad P^L_{\mu\nu}P^{L\nu\lambda} = P^{L\lambda}_{\mu}, \quad P^T_{\mu\nu}P^{L\nu\lambda} = 0, \quad P^T_{\mu\nu} + P^L_{\mu\nu} = \eta_{\mu\nu} \, . \tag{4}$$

c) A general property of the propagator is that it is a Green's function, i.e. it is the inverse of the operators appearing in the quadratic terms in a lagrangian. In momentum space and in case of the abelian theory (2), the propagator $\tilde{\Delta}_{\mu\nu}(p)$ can be defined via

$$\tilde{\Delta}_{\mu\nu}(p)\,\mathcal{O}^{\nu\lambda}(p) = i\delta^\lambda_\mu \, . \tag{5}$$

Show that there exists no inverse for the operator $\mathcal{O}^{\nu\lambda}(p)$ by making a general ansatz for $\tilde{\Delta}_{\mu\nu}(p)$ using $P^T_{\mu\nu}$ and $P^L_{\mu\nu}$. This is one way to see that a sensible definition of the quantum theory requires gauge fixing by adding a Lagrangian multiplier, such that

$$\mathcal{S}_{0,\xi} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2\xi}\Big(\partial_\mu A^\mu(x)\Big)^2\right], \tag{6}$$

where $\xi \in \mathbb{R}$ is the gauge fixing parameter. Calculate the corresponding operator $\mathcal{O}^\xi_{\mu\nu}(p)$ in momentum space and show that it is well-defined and reads

$$\tilde{\Delta}^\xi_{\mu\nu}(p) = \left(-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\frac{i}{p^2} \, . \tag{7}$$

d) Now consider a massive spin-1 particle by adding a mass term to the original action (1), such that

$$\mathcal{S}_{0,m} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}m_A^2 A_\mu(x)A^\mu(x)\right], \tag{8}$$

and derive the corresponding propagator $\tilde{\Delta}^m_{\mu\nu}(p)$ in momentum space.

2. Solutions to the Dirac equation (3+2+2 points)

For an antiparticle in the rest frame, i.e. with momentum $p_R = (m, \vec{0}^T)$ and mass m, the general solution to the free Dirac equation reads

$$v(p_R, s) = \sqrt{m} \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}, \tag{9}$$

where η_s is a two-component spinor of the rotation group describing the spin-orientation of the Dirac solution, and is normalized to $\eta_s^\dagger \eta_s = 1$ for $s \in \{1, 2\}$. In the following, use the chiral (Weyl) representation of the Dirac matrices.

a) First, check that (9) solves the Dirac equation for an antiparticle in momentum space. Now consider a boost along the 3-direction with rapidity λ, which transforms the 4-momentum vector to

$$\begin{pmatrix} p^0 \\ p^3 \end{pmatrix} = \exp\left[\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} p_R^0 \\ p_R^3 \end{pmatrix}, \tag{10}$$

while leaving the transversal components p_R^1 and p_R^2 invariant. Review the transformation behavior of a spinor as given in the lecture and calculate

$$v(p, s) = \exp\big(-i\lambda\, J^{03}\big) v(p_R, s) \qquad \text{mit} \qquad J^{03} = -\frac{i}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}. \tag{11}$$

You should arrive at the following result

$$v(p, s) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \eta_s \\ -\sqrt{p \cdot \bar{\sigma}}\, \eta_s \end{pmatrix}, \qquad u(p, s) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi_s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi_s \end{pmatrix}, \tag{12}$$

where the solution for the particle spinor $u(p, s)$ was added for the sake of completeness.

b) Verify the normalization properties

$$\bar{u}(p, r)\, u(p, s) = 2m\, \delta_{rs}\,, \qquad \bar{v}(p, r)\, v(p, s) = -2m\, \delta_{rs}\,, \tag{13}$$

and the completeness relations

$$\sum_{s=1}^{2} u(\vec{p}, s)\, \bar{u}(\vec{p}, s) = \not{p} + m\,, \qquad \sum_{s=1}^{2} v(\vec{p}, s)\, \bar{v}(\vec{p}, s) = \not{p} - m, \tag{14}$$

while using that $\xi_r^\dagger \xi_s = \delta_{rs} = \eta_r^\dagger \eta_s$ and $\sum_{s=1,2} \xi_s \xi_s^\dagger = 1 = \sum_{s=1,2} \eta_s \eta_s^\dagger$.

c) Now we switch to the helicity basis for the 2-component spinors, that describe the spin orientation. We introduce $\xi_\pm$ and $\eta_\pm$ that are eigenspinors of the helicity operator $h \equiv \frac{\vec{p} \cdot \vec{\sigma}}{2|\vec{p}|}$. A convenient ansatz for the solutions to the Dirac equation is given by

$$u(p, \pm) = \sqrt{\frac{p^0 + m}{2}} \begin{pmatrix} \lambda_\pm\, \xi_\pm \\ \lambda_\mp\, \xi_\pm \end{pmatrix}, \qquad v(p, \pm) = \sqrt{\frac{p^0 + m}{2}} \begin{pmatrix} \lambda_\mp\, \eta_\pm \\ -\lambda_\pm\, \eta_\pm \end{pmatrix}, \tag{15}$$

where $\lambda_\pm = (p^0 + m \mp |\vec{p}|)/(p^0 + m)$. Determine the eigenvalues to $\xi_\pm$, $\eta_\pm$ of the helicity operator by demanding that the spinors in (15) fulfill the corresponding Dirac equations in momentum space. Then consider the massless limit $m \to 0$. What is the relationship between chirality and helicity for a particle and an antiparticle?

3. The Lorentz algebra (3+3 points)

a) Given the defining relations for the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, show that the matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \tag{16}$$

satisfy the algebra of the generators of the Lorentz group, i.e.

$$[S^{\mu\nu}, S^{\alpha\beta}] = i\left(\eta^{\mu\beta}S^{\nu\alpha} + \eta^{\nu\alpha}S^{\mu\beta} - \eta^{\mu\alpha}S^{\nu\beta} - \eta^{\nu\beta}S^{\mu\alpha}\right) \ . \tag{17}$$

b) Show the following transformation property of the Dirac matrices

$$D^{-1}(\Lambda)\,\gamma^\mu\,D(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu, \tag{18}$$

under proper and orthochronous Lorentz transformations $\Lambda \in L_+^\uparrow$ by considering an infinitesimal transformation $\Lambda^\mu{}_\nu = \eta^\mu_\nu + w^\mu{}_\nu$, where $|w^\mu{}_\nu| \ll 1$ for each entry. Remember that a global Lorentz transformation in spinor space is given by $D(\Lambda) = \exp\left(-\frac{i}{2}w_{\mu\nu}S^{\mu\nu}\right)$.

Please write down how long it took you to solve the exercises.