Quantenfeldtheorie und Theoretische Elementarteilchenphysik
(exercise sheet 8)

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1 The Master Formula

The goal of the following problem is to derive the master formula $(a, b > 0)$

$$I_d(a,b;\Delta) = \int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)^a}{(l_E^2+\Delta)^b} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\frac{d}{2}+a)\Gamma(b-\frac{d}{2}-a)}{\Gamma(\frac{d}{2})\,\Gamma(b)} \left(\frac{1}{\Delta}\right)^{b-\frac{d}{2}-a}, \quad (1)$$

for momentum integrals in a d-dimensional Euclidean space $(d > 0)$, where $l_E^2 = \sum_{i=1}^d l_i^2$ is the square of the length of the d-dimensional momentum vector. Assume that $b > a+d/2$ and $\Delta \neq 0$, so that the integral converges as $l_E^2 \to \infty$ and $l_E^2 \to 0$.

1.1 [2 points]

Factorize the d-dimensional measure as

$$d^d l_E = d\Omega_{d-1}(l_E^2)^{d/2-1}\frac{d(l_E)^2}{2}, \quad (2)$$

where $d\Omega_{d-1}$ is the integration measure for the $(d-1)$ angular coordinates of the sphere $S_{d-1} = \{x \in \mathbb{R}^d : \sum_{i=1}^n x_i^2 = 1\}$, and l_E^2 with $0 \leq l_E^2 < \infty$ is the square of the radial coordinate. Use the fact that $\sqrt{\pi} = \int_{-\infty}^{+\infty} dx e^{-x^2}$ to express $\pi^{d/2}$ as an integral over d-dimensional Euclidian space, and evaluate the radial integral using the definition

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}\, e^{-t}, \quad (3)$$

of the Γ-function. Note that for integer argument $\Gamma(n) = (n-1)!$, and that in general $\Gamma(1+z) = z\Gamma(z)$. Calculate the area $\int d\Omega_{d-1}$ of the sphere S_{d-1} (i.e. the surface of the ball $B_d = \{x \in \mathbb{R}^d : \sum_{i=1}^n x_i^2 \leq 1\}$).

1.2 [3 points]

Evaluate the integral $I_d(a,b;\Delta)$ using the substitution $l_E^2/\Delta = (1-x)/x$ and the definition

$$B(z,w) = \int_0^1 dx\, x^{z-1}\,(1-x)^{w-1} = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad (4)$$

of the Euler β-function, and derive the master formula (1).

1.3 [3 points]

The beauty of the master formula is that, by analytic continuation, it can be used to define the value of the integral $I_d(a,b;\Delta)$ for arbitrary values of the exponents a and b, as well as for an arbitrary value of the space-time dimension d. Note that the function $\Gamma(z)$ is analytic in the complex z plane except for poles at $z = 0, -1, -2, \ldots$. The divergent behaviour of the original integral is reflected in the presence of poles for specific values of a, b, and d. Choosing a value for d different from the physical value $d = 4$ is by far the most convenient way of regularizing divergent momentum integrals. This method is called *dimensional regularisation* and was invented by the Nobel Laureates G. 't Hooft and M. Veltman.

Consider, as an example, the case where $a = 0$ and $b = 2$. The integral $I_4(0, 2; \Delta)$ in 4 dimensions is logarithmically divergent for $l_E^2 \to \infty$, as indicated by the pole of the factor $\Gamma(b - a - d/2)$ in the master formula. However, the integral converges in a space with less than 4 dimensions. Derive the expression for $I_d(0, 2; \Delta)$ assuming that $d < 4$. Now set $d = 4 - 2\epsilon$ and consider $\epsilon > 0$ as an infinitesimal parameter, never mind that we have little intuition about the geometry of a space with non-integer dimensionality. Derive an expression for the product $\mu^{4-d} I_d(0, 2; \Delta)$ as a Laurent series in ϵ, dropping terms that vanish in the limit $\epsilon \to 0$ (the factor μ^{4-d} is inserted so as to make the result dimensionless). Use the expansion $\Gamma(\epsilon) = 1/\epsilon - \gamma_E + \mathcal{O}(\epsilon)$, where $\gamma_E = 0.577\ldots$ is Euler's constant, and introduce the notation $1/\hat{\epsilon} = 1/\epsilon - \gamma_E + \ln(4\pi)$ to simplify the answer.

2 Independence of the regularisation scheme [6 points]

Let us consider the logarithmically divergent momentum integral

$$\Pi(s) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - s + i\epsilon)^2}, \tag{5}$$

that can arise, e.g. from a 1-loop diagram with two massive bosonic propagators and external momenta set to zero. In order to work with a well-defined integral we can regularise it by introducing a hard momentum-cutoff (Pauli-Villars regularisation) and define

$$\Pi_\Lambda(s) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - s + i\epsilon)^2} \frac{-\Lambda^2}{k^2 - \Lambda^2 + i\epsilon}, \tag{6}$$

or by allowing for a deviation in the space-time dimension $d = 4 - 2\epsilon$ with $\epsilon > 0$ (dimensional regularisation)

$$\Pi_d(s) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - s + i\epsilon)^2}. \tag{7}$$

Now, we consider the following renormalised integrals

$$\Pi_1^{\text{ren}}(s) \equiv \lim_{\Lambda\to\infty} \Big[\Pi_\Lambda(s) - \Pi_\Lambda(s_0)\Big], \qquad \Pi_2^{\text{ren}}(s) \equiv \lim_{d\to 4} \Big[\Pi_d(s) - \Pi_d(s_0)\Big], \tag{8}$$

that are defined in order to respect the renormalisation condition $\Pi^{\text{ren}}(s_0) = 0$ at the scale s_0. Calculate Π_1^{ren} and Π_2^{ren} and show that both are identical.