Quantenfeldtheorie und Theoretische Elementarteilchenphysik

(exercise sheet 7)

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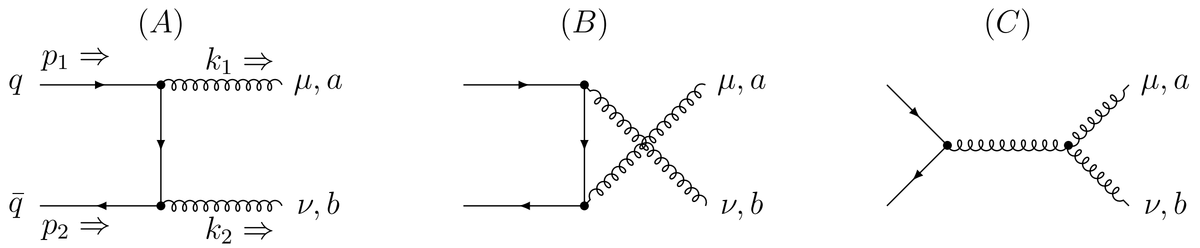
1 Checking gauge invariance in QCD

Consider the process $q\bar{q} \to gg$ of the production of two gluons from the annihilation of a quark and an anti-quark, the contributing diagrams at leading order can be seen below.



Here p_1, p_2 are the momenta of the incoming quark, anti-quark with $p_i^2 = m^2$ and $a, b = 1, ..., 8$ are the color indices of the outgoing gluons with momenta k_1, k_2 and $k_i^2 = 0$.

1.1 [6 points]

First write down the amplitudes $(M_A^{\mu\nu})_{ab}$ and $(M_B^{\mu\nu})_{ab}$ for diagrams (A) and (B). The physical amplitude can be obtained by adding them up to

$$M_{ab}^{A+B} = \Big[(M_A^{\mu\nu})_{ab} + (M_B^{\mu\nu})_{ab}\Big]\, \epsilon_\mu^\star(k_1,\lambda_1)\, \epsilon_\nu^\star(k_2,\lambda_2), \tag{1}$$

where $\epsilon_\mu(k_i, \lambda_i)$ are the gluon polarizations vectors. For physical gluons those are transverse, i.e.

$$k_i^2 = 0, \qquad \lambda_i = \pm 1, \qquad k_i^\mu\, \epsilon_\mu(k_i,\lambda_i) = 0\ . \tag{2}$$

Gauge invariance implies that we are allowed to shift e.g. $\epsilon_\mu(k_1, \lambda_1) \to \epsilon_\mu(k_1, \lambda_1) - \alpha\, k_{1\mu}$, where α is some constant number. In order to check the principle of gauge invariance calculate

$$k_{1\mu}\left[(M_A^{\mu\nu})_{ab} + (M_B^{\mu\nu})_{ab}\right]\ . \tag{3}$$

Explain why the result is not zero and compare to the case of QED.

1.2 [4 points]

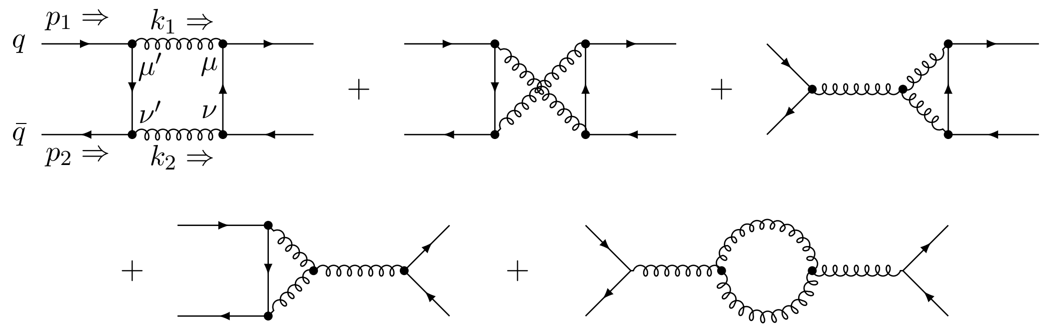
Now calculate the amplitude $(M_C^{\mu\nu})_{ab}$ for diagram (C) using the Feynman-'t Hooft gauge $\xi = 1$ for the gluon propagator. Check again the result this time for

$$k_{1\mu}\left[(M_A^{\mu\nu})_{ab} + (M_B^{\mu\nu})_{ab} + (M_C^{\mu\nu})_{ab}\right]\ . \tag{4}$$

Show that if in addition the ν index is contracted with the transverse gluon polarization vector $\epsilon_\nu(k_2, \lambda_2)$ the expression vanishes.

2 Unitarity and ghosts

This exercise will demonstrate that we need ghosts to preserve unitarity in QCD. For that purpose we consider the example of a loop process $q\bar{q} \to q\bar{q}$ with 2 gluon (virtual) intermediate states. The following diagrams contribute to the process at one-loop order.



From the unitarity condition of the S-matrix $S^\dagger S = \mathbf{1}$ one can derive a relation for the elastic scattering process where the initial and final states are the same. Defining the scattering amplitude T via $S = 1 - iT$ one can derive the relation (no summation over i)

$$2\,\mathrm{Im}\,T_{ii} = -\sum_n (T^\dagger)_{in}\,(T)_{ni}\,, \tag{5}$$

that must be valid if the total probability is conserved. Here T_{ni} means the scattering amplitude for the process of a state $|i\rangle$ scattering into a state $|n\rangle$. In our case the sum over intermediate states $|n\rangle$ involves two gluon states, while the state $|i\rangle$ is composed of a quark and an anti-quark. The right- (RHS) and the left-(LHS) hand side can be calculated to

$$\mathrm{RHS} = -\frac{1}{2}\int d\Omega^{(2)}\, T^{\dagger\mu\nu}_{ab}\,(-g_T)_{\mu\mu'}\,(-g_T)_{\nu\nu'}\,T^{\mu'\nu'}_{ab} \tag{6}$$

$$\mathrm{LHS} = -\frac{1}{2}\int d\Omega^{(2)}\, T^{\dagger\mu\nu}_{ab}\,(-g_{\mu\mu'})\,(-g_{\nu\nu'})\,T^{\mu'\nu'}_{ab}, \tag{7}$$

where $T^{\mu\nu}_{ab} = i(M^{\mu\nu}_A + M^{\mu\nu}_B + M^{\mu\nu}_C)_{ab}$ with the amplitudes that have been calculated in the previous exercise task 1. Here, $\Omega^{(2)}$ is the phase-space for two gluons in the intermediate states. The transverse projector involves only transverse gluon polarizations and is defined by

$$(-g_T)_{\mu\mu'} \equiv \sum_{\lambda_1=\pm1} \epsilon_\mu(1)\,\epsilon^*_{\mu'}(1), \qquad (-g_T)_{\nu\nu'} \equiv \sum_{\lambda_2=\pm1} \epsilon_\nu(2)\,\epsilon^*_{\nu'}(2), \tag{8}$$

where $\epsilon_\mu(i) \equiv \epsilon_\mu(k_i, \lambda_i)$. In case of two gluons we can add scalar and longitudinal polarizations by introducing the Sudakov unit-vectors $\hat{k}_1, \hat{k}_2$ such that the metric can be written as

$$-g^{\mu\mu'} = (-g_T)^{\mu\mu'} - \frac{1}{2}\Big(\hat{k}^\mu_1\,\hat{k}^{\mu'}_2 + \hat{k}^{\mu'}_1\,\hat{k}^\mu_2\Big), \quad -g^{\nu\nu'} = (-g_T)^{\nu\nu'} - \frac{1}{2}\Big(\hat{k}^\nu_1\,\hat{k}^{\nu'}_2 + \hat{k}^{\nu'}_1\,\hat{k}^\nu_2\Big), \tag{9}$$

with

$$\hat{k}^\mu_1 \equiv \sqrt{\frac{2}{k_1\cdot k_2}}\,k^\mu_1, \qquad \hat{k}^\mu_2 \equiv \sqrt{\frac{2}{k_1\cdot k_2}}\,k^\mu_2\,. \tag{10}$$

2.1 Unitarity in general [2 points]

Derive relation (5). Start from the unitarity condition $S^\dagger S = \mathbf{1}$ and consider the scattering of an initial state $|i\rangle$ into a final state $|f\rangle$. Insert then a complete set of intermediate states $\sum_n |n\rangle\langle n|$. Finally specialize to the case $|i\rangle = |f\rangle$.

2.2 [1 point]

Check the first equation in (9) by contracting both sides with $(k_1)_\mu$.

2.3 Check unitarity for $q\bar{q} \to q\bar{q}$ [4 points]

Next we can calculate the difference

$$\Delta T \equiv \text{RHS} - \text{LHS}, \tag{11}$$

where you can simplify the expression by using the properties of the amplitudes $T^{\mu\nu}_{ab}$ when contracted with the gluon momenta and polarizations. Only terms coming from the longitudinal part of $g_{\mu\mu'}$ and $g_{\nu\nu'}$ remain. Show with the results from the previous exercise (use your own results in case you have other sign conventions)

$$\hat{k}_{1\mu'}\, T^{\mu'\nu'}_{ab} = ig^2\, f_{abc}\, T_c\, \bar{v}(p_2, s_2)\, \frac{\hat{k}_2^{\nu'} \not{k}_1}{2k_1 \cdot k_2} u(p_1, s_1)\,, \tag{12}$$

$$\hat{k}_{2\nu'}\, T^{\mu'\nu'}_{ab} = -ig^2\, f_{abc}\, T_c\, \bar{v}(p_2, s_2)\, \frac{\hat{k}_1^{\mu'} \not{k}_2}{2k_1 \cdot k_2} u(p_1, s_1)\,, \tag{13}$$

where T_c is a generator of $SU(3)$ and f_{abc} are the structure constants, that the remaining terms can be expressed by

$$\Delta T = \int d\Omega^{(2)}\, T^\dagger_{ab} T_{ab}, \tag{14}$$

with

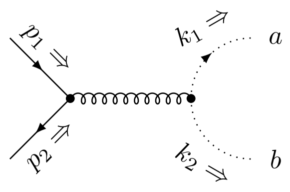
$$T_{ab} \equiv ig^2 f_{abc}\, T_c\, \bar{v}(p_2, s_2)\, \frac{\not{k}_1 - \not{k}_2}{4k_1 \cdot k_2}\, u(p_1, s_1)\,. \tag{15}$$

This would mean that unitarity is violated by the diagrams with 2 longitudinally polarized intermediate gluons.

2.4 Restoration of unitarity [3 points]

We can add extra unphysical fields (ghosts, represented by dotted lines) so that they restore unitarity. Use the Feynman rule for ghosts and calculate the following diagram

$$-i\tilde{T}_{ab} \quad =$$



Then you can check that the ghost contribution

$$\int d\Omega^{(2)}\, \tilde{T}^\dagger_{ab} \tilde{T}_{ab}\,, \tag{16}$$

to the LHS of (5) precisely cancels the remaining scalar and longitudinal contributions from the gluon in (14).