Quantenfeldtheorie und Theoretische Elementarteilchenphysik

(exercise sheet 6)

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1 Ward identities in QED

The generating functional for quantum electrodynamics (QED) in R_ξ gauge is of the form

$$Z[\eta, \bar{\eta}, J] = N \int \mathcal{D}A_\mu \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left[i \int d^4x \, \mathcal{L}_{\text{eff}} \right], \tag{1}$$

where N is a normalization factor and with the Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} i (\partial_\mu + ieA_\mu) \gamma^\mu \psi - m\bar{\psi}\psi - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi, \tag{2}$$

where ξ is the gauge-fixing parameter.

1.1 (4 points)

Show that if we require that the generating functional $Z[\eta, \bar{\eta}, J]$ is to be invariant under an infinitesimal $U(1)$ gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \qquad \text{and} \qquad \psi(x) \to \psi(x) - ie\alpha(x)\psi(x), \tag{3}$$

we get the following relation

$$\left[-\frac{1}{\xi} \Box \, \partial^\mu \frac{\delta}{\delta J^\mu} - i\partial_\mu J^\mu + ie \left(\eta \frac{\delta}{\delta \eta} - \bar{\eta} \frac{\delta}{\delta \bar{\eta}} \right) \right] Z[\eta, \bar{\eta}, J] = 0, \tag{4}$$

which is called the Ward-Takahashi identity. *Remark: be careful with the anti-commuting nature of the fermion fields and sources.*

1.2 (4 points)

Translate the identity (4) into an equation of the generating functional for the connected Green's functions $W[\eta, \bar{\eta}, J]$ by $Z = e^{iW}$. Then we define a new functional $\Gamma[\bar{\psi}, \psi, A_\mu]$ as the Legendre transform of $W[\eta, \bar{\eta}, J_\mu]$

$$\Gamma[\bar{\psi}, \psi, A_\mu] = W[\eta, \bar{\eta}, J_\mu] - \int d^4x \, (J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi), \tag{5}$$

where

$$\psi(x) = \frac{\delta W}{\delta \bar{\eta}(x)}, \qquad \bar{\psi}(x) = -\frac{\delta W}{\delta \eta(x)}, \qquad A_\mu = \frac{\delta W}{\delta J^\mu(x)}, \tag{6}$$

are usually called the classical fields. Express the Ward-Takahashi identity in terms of the functional $\Gamma[\bar{\psi}, \psi, A_\mu]$.

1.3 (4 points)

Now differentiate the result of part (1.2) with respect to $\bar{\psi}(x_1)$ and $\psi(x_2)$ and then set $\psi = \bar{\psi} = A_\mu = 0$ afterwards. You should obtain

$$\partial_\mu^{x_3} \frac{\delta}{\delta\psi(x_2)} \frac{\delta}{\bar{\psi}(x_1)} \frac{\delta}{\delta A_\mu(x_3)} \Gamma[0,0,0] = ie\Big[\delta(x_3 - x_2) - \delta(x_3 - x_1)\Big] \frac{\delta}{\delta\psi(x_2)} \frac{\delta}{\delta\bar{\psi}(x_1)} \Gamma[0,0,0],$$

where the notation is such that you first perform the functional derivatives and then set $\bar{\psi} = \psi = A_\mu = 0$. Then transform this expression to momentum space by multiplying it with $\exp[-i(px_1 + qx_3 - p'x_2)]$ and integrating it over the position coordinates. You should obtain the familiar form of the Ward identity,

$$q^\mu \, \Gamma_\mu(p, q, p+q) = \Big[\tilde{S}_F(p+q)\Big]^{-1} - \Big[\tilde{S}_F(p)\Big]^{-1}, \tag{7}$$

where the proper vertex function Γ_μ and the dressed propagator $\tilde{S}_F$ are related to the functional Γ by

$$\int d^4x_1 d^4x_2 d^4x_3 \, e^{-i(px_1 + qx_3 - p'x_2)} \frac{\delta^3\Gamma[0,0,0]}{\delta\psi(x_2)\delta\bar{\psi}(x_1)\delta A_\mu(x_3)} = -e\,(2\pi)^4 \,\delta^4(p + q - p')\,\Gamma^\mu(p, q, p')$$

$$\int d^4x_1 d^4x_2 e^{-i(px_1 - p'x_2)} \frac{\delta^2\Gamma[0,0,0]}{\delta\psi(x_2)\delta\bar{\psi}(x_1)} = (2\pi)^4 \,\delta^4(p - p')\,\Big[\tilde{S}_F(p)\Big]^{-1}.$$

Then draw a diagrammatic representation of the Ward-Takahashi identity (7).

1.4 (1 point)

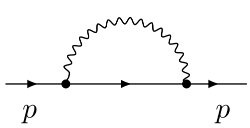
Check the Ward-Takahashi identity (7) at leading order (tree-level). Note that $\Gamma_\mu(p, q, p+q)$ is defined without the electromagnetic charge e.

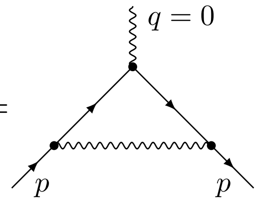
1.5 (7 points)

Now we want to check the identity at next-to-leading order (one-loop level). For this purpose we stick to the case of vanishing photon momentum $q = 0$. In this case (7) can be expressed in the following form

$$\Gamma_\mu(p, 0, p) - \gamma_\mu = -\frac{\partial\Sigma(p)}{\partial p^\mu}, \tag{8}$$

where $\Sigma(p)$ contains only 1 particle-irreducible diagrams for the self-energy. At next-to-leading order you need to calculate the two amputated (without external propagators) amplitudes below (you do not need to perform the loop momentum integration), where an abbreviation is defined $\Lambda_\mu(p, 0, p) \equiv \Gamma_\mu(p, 0, p) - \gamma_\mu$.

$\Sigma(p) =$ 

$\Lambda_\mu(p, 0, p) =$ 

Note that you can work in the Feynman-'t Hooft gauge ($\xi = 1$) for the photon propagators. Then you can compare both amplitudes and check (8).