Quantenfeldtheorie und Theoretische Elementarteilchenphysik

(exercise sheet 5)

Tobias Hurth, Andreas v. Manteuffel & Raoul Malm SS 2014

Questions: malmr@uni-mainz.de Hand-in on June 3rd, 2014

1 Grassmann variables

In an n-dimensional Grassmann algebra the n generators $x_1, x_2, x_3, ..., x_n$ satisfy

$$\{x_i, x_j\} = 0 \qquad i, j = 1, 2, 3, ..., n \tag{1}$$

and every element can be expanded in a finite series

$$f(x) = f^{(0)} + f^{(1)}_{i_1} x_{i_1} + f^{(2)}_{i_1,i_2} x_{i_1} x_{i_2} + \cdots + f^{(n)}_{i_1,...,i_n} x_{i_1} \dots x_{i_n} \tag{2}$$

where each of the summed-over indices $i_1, i_2, ..., i_n$ ranges from 1 to n. The expansion terminates because of (1). To perform integrations we need the symbols $dx_1, dx_2, ..., dx_n$ which are introduced to satisfy the conditions

$$\{dx_i, dx_j\} = 0, \qquad \int dx_i = 0, \qquad \int dx_i\, x_j = \delta_{ij}\ . \tag{3}$$

1.1 [3 points]

Let us change the integration variable to $x'_i = M_{ij} x_j$, where x, x' are Grassmann-valued vectors with $x^T = (x_1, x_2, ..., x_n)$, $x'^T = (x'_1, x'_2, ..., x'_n)$ and where M is a $n \times n$ matrix with $\det M \neq 0$. Show that this change of integration variables leads to

$$\int dx'_n \dots dx'_1\, f(x') = \int dx_n \dots x_1 \left[\det M\right]^{-1} f(x'(x))\ . \tag{4}$$

1.2 [3 points]

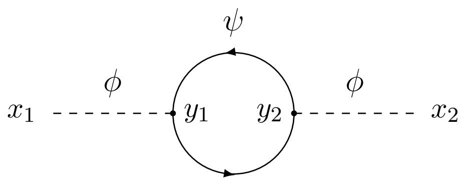
Next evaluate the Gaussian integral

$$G(A) = \int dy_n dx_n \dots dy_1 dx_1\ \exp\left[x^T A\, y\right], \tag{5}$$

where x, y are Grassmann-valued vectors with $x^T = (x_1, x_2, ..., x_n)$, $y^T = (y_1, y_2, ..., y_n)$ and A is a $n \times n$ matrix with $\det A \neq 0$.

2 Minus sign for a fermion loop

In this exercise we want to show that a fermion loop leads to an additional minus sign relative to the hypothetical case if fermions would obey Bose-Einstein statistics. As an example consider the loop diagram with two external scalar fields, as shown below.



The scalar-fermion-fermion interaction is given by

$$\mathcal{L}_{\text{int}}(\bar{\psi}, \psi, \phi) = -g\, \phi\, \bar{\psi}\psi, \tag{6}$$

where g is a dimensionless coupling. The generating functional reads

$$Z[\eta, \bar{\eta}, J] = \exp\left[i \int d^4x\, \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta\eta(x)}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}, \frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_0[\eta, \bar{\eta}]\, Z_0[J], \tag{7}$$

with

$$Z_0[\eta, \bar{\eta}] = \exp\left[-i \int d^4z_1 \int d^4z_2\, \bar{\eta}(z_1)\, S(z_1, z_2)\, \eta(z_2)\right], \tag{8}$$

$$Z_0[J] = \exp\left[-\frac{i}{2} \int d^4z_1 \int d^4z_2\, J(z_1)\, \Delta_F(z_1, z_2)\, J(z_2)\right], \tag{9}$$

where $S(z_1, z_2)$, $\Delta_F(z_1, z_2)$ are the fermion and scalar propagators.

2.1 [1 point]

In a first step convince yourself (no calculation needed) that the relevant diagram is contained in the scalar two-point Green's function at order g^2 and can be written as

$$\begin{aligned} G_\phi(x_1, x_2) \ni -g^2 \int d^4y_1\, d^4y_2 &\left(\frac{\delta^2}{\delta J(x_1)\, \delta J(x_2)} \frac{\delta}{\delta J(y_1)} \frac{\delta}{\delta J(y_2)}\, Z_0^{(2)}[J]\right) \\ &\times \left(\frac{\delta^2}{\delta\eta_a(y_1)\, \delta\bar{\eta}_a(y_1)} \frac{\delta^2}{\delta\eta_b(y_2)\, \delta\bar{\eta}_b(y_2)}\, Z_0^{(2)}[\eta, \bar{\eta}]\right), \end{aligned} \tag{10}$$

where a, b are spinor indices. The expressions $Z_0^{(2)}[J]$ and $Z_0^{(2)}[\eta, \bar{\eta}]$ are the second-order expansions of the exponentials of (9) and (8). Write them down.

2.2 [3 points]

Now compute the functional derivatives acting on $Z_0^{(2)}[\eta, \bar{\eta}]$, i.e. only the second bracket in (10), which shows that there appears an additional minus sign.