Quantenfeldtheorie und Theoretische Elementarteilchenphysik
(exercise sheet 3)

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1 Path integral for the harmonic oscillator

Consider the one-dimensional harmonic oscillator with the Lagrangian

$$L(q(t), \dot{q}(t)) = \frac{m}{2}\dot{q}^2(t) - \frac{mw^2}{2}q^2(t), \tag{1}$$

with mass $m > 0$ and angular frequency $w > 0$.

1.1 (10 points)

Calculate the transition amplitude (set $\hbar = 1$)

$$K(q_f, T; q_i, 0) \equiv \langle q_f, T | q_i, 0 \rangle = \int_{q(0)=q_i}^{q(T)=q_f} \mathcal{D}q \, \exp\left[i \int_0^T dt L(q(t), \dot{q}(t)) \right], \tag{2}$$

for $\frac{\pi}{w} > T > 0$ in the path-integral formalism. The boundary conditions are $q(0) = q_i$ and $q(T) = q_f$. You should find the following result

$$K(q_f, T; q_i, 0) = \left(\frac{mw}{2\pi i \sin(wT)} \right)^{1/2} \exp\left(\frac{imw}{2\sin(wT)} \left[(q_i^2 + q_f^2)\cos(wT) - 2q_i q_f \right] \right). \tag{3}$$

Remark: One suggestion to solve the problem is to use that an arbitrary path can be written as $q(t) = q_c(t) + \eta(t)$, where $q_c(t)$ is the classical path fulfilling the Euler-Lagrange equations of (1) with the corresponding boundary conditions $q_c(0) = q_i$, $q_c(T) = q_f$, while $\eta(t)$ is defined to be a deviation from that path. You should obtain

$$K(q_f, T; q_i, 0) = \exp[iS[q_c]] \int_{\eta(0)=0}^{\eta(T)=0} \mathcal{D}\eta \, \exp[iS[\eta]], \tag{4}$$

where $S[q_c]$ is the action when the classical solution is inserted, while $S[\eta]$ is the action inserting the deviations $\eta(t)$. To perform the path integral one can expand $\eta(t)$ into orthonormal basis functions such that

$$\eta(t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{T}} \sin\left(\frac{n\pi t}{T} \right), \tag{5}$$

with coefficients c_n. Then you can first perform the t-integration in the action $S[\eta]$. After that you can discretize the path integral into $N+1$ equal segments and express the measure as

$$\mathcal{D}\eta = \lim_{N\to\infty} C_N \int \left(\prod_{n=1}^{N} dc_n \right), \tag{6}$$

where C_N is some normalization constant (independent of w). This will allow you to perform the integrations. Finally you can determine the normalization constant by setting $w \to 0$ and comparing your expression with the result for the freely moving particle.

1.2 (5 points)

Check whether the obtained amplitude (3) obeys the Schrödinger equation,

$$i\frac{\partial}{\partial t}K(q_f,t;q_i,0) = H\,K(q_f,t;q_i,0), \tag{7}$$

where H is the Hamiltonian corresponding to the Lagrangian in (1) expressed in terms of q_f and $p_f = -i\,\partial/\partial q_f$. Then find the energy spectrum of K and show that it takes the form $E_n = w(n+1/2)$, with $n \in \mathbb{N}$.